Analysis of sets of convex bodies in 3D space with Minkowski functionals

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ABSTRACT

Convex bodies are frequently used as atoms of more complex objects in computer vision. The aim of this paper is to present the Minkowski functionals as a tool for convex shape analysis for single bodies. We define ratios of these functionals to elaborate a mapping from 3D bodies into the unit square. Thus we can discriminate between filaments, pancakes and ribbons.

Keywords: Minkowski functionals, isoperimetric inequality, mathematical morphology, shape representation, shape analysis, convexity.

1. INTRODUCTION

Convex bodies play a significant role in shape description and analysis, especially in fields such as computer vision, graphics and image processing.[1] The most frequently adopted technique in dealing with a complex geometric object is to approximate it by turning out into a convex object (a rectangular box or an ellipsoid) or to decompose it into a union of convex constituents. In mathematics the most widely used representation scheme for convex bodies is the support function representation introduced by Minkowski in the beginning of the XXth century.[2] Simple algebraic operations on support functions result in a variety of geometric operations on the corresponding geometric objects.

An another point of view in the study of convex bodies is the using of a family of functionals, called Minkowski functionals. By definition, a functional is a global parameter associated with a set, i.e. a mapping from $\mathcal{P}(\mathbb{R}^d)$ onto $\mathbb{R}$. Integral geometry[3] provides a rigourous mathematical framework to define these functionals.

When Matheron and Serra[4] funded the mathematical morphology they introduced the notion of morphology opening associated with the convexity of the structuring element ; thus they rediscovered the Minkowski functionals. However, at our best knowledge these functionals have seldom been used in the strict field of image processing.[5] Yet the Minkowski functionals have been recently applied with success to fields as varied as astrophysics[6 7] or the polymer characterization,[8] thus showing their versatility.

In this paper we present a mapping, called Blaschke diagram, of the family of compact convex sets of $\mathbb{R}^3$ to a point in $\mathbb{R}^2$. This mapping uses a combinaison of Minkowski functionals and allows us to describe the geometry and topology of individual objects.

2. CONVEX SETS AND STEINER'S FORMULA

In this paper we will work exclusively in the linear space $\mathbb{R}^d$, where $d = 0, 1$, or 2. The letters $x, y, \ldots$ are used to denote points as well as vectors. The sum $x + y$ of the points $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ coincides with the sum of two vectors $x + y = (x_1 + y_1, \ldots, x_d + y_d)$, and the scalar multiplication of a point x with a real number c is defined as $cx = (cx_1, \ldots, cx_d)$. Thus the difference $y - x$ can be obtained from $y + cx$ for $c = -1$. The distance between two points x and y is the norm of their difference, $\|y - x\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_d - x_d)^2}$. A subset K of $\mathbb{R}^d$ is said convex if for every x and y in K the segment $[x, y]$ of $\mathbb{R}^d$, whose ends are x and y, is all contained in $K : [x, y] \subset K$. A closed bounded convex set is called a convex body. The convex sets of $\mathbb{R}$ are the intervals. Given a point x in $\mathbb{R}^d$ and a real positive number r, the set

$$b(x, r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$$

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is called a ball or a d-dimensional solid sphere with center x and radius r. Its interior $\{y \in \mathbb{R}^d : \|y - x\| < r\}$ is an open ball and its boundary $\{y \in \mathbb{R}^d : \|y - x\| = r\}$ a sphere. The volume of the unit sphere is denoted by ω_d ($\omega_0 = 1, \omega_1 = 2$, $\omega_2 = \pi$, $\omega_3 = 4\pi/3$. Two immediate properties of convex sets are the following :

- If K is a plate in $\mathbb{R}^2$ or a body in $\mathbb{R}^3$, of some material, then it contains its center of mass (center of gravity) ; constant density is not required.
- The property of convexity for open convex sets is invariant by continuous deformation. All open convex sets of $\mathbb{R}^d$ are equivalent, in particular equivalent to $\mathbb{R}^d$ itself.

The Minkowski addition is a set transformation which changes the size and the shape of a given set. It is carried out as follows : independently of a given set A, we choose another set B which we call the structuring element. The Minkowski addition of A and B, written $A \oplus B$, translates, enlarges, and deforms the set A. It is defined by

$$A \oplus B = \{x + y : x \in A, y \in B\} \tag{1}$$

The above definition is equivalent to the relations

$$A \oplus B = \bigcup_{y \in B} (A + y), \qquad A \oplus B = \bigcup_{x \in A} (B + x) \tag{2}$$

Obviously, the Minkowski sum of two convex bodies is itself convex. If the set B is a ball of radius r then the Minkowski sum is said to be a parallel set of A. This new set is also called A dilated by the ball of radius r ; it is the set of points whose distance to A is at most r. In order to characterize a body A one looks for the dilated set by a ball of radius r. Let us take some examples. The new sets for a point in a plane, a line in a plane and a rectangle in a plane (Figure 1) are respectively :

1. a disk of radius r,
2. the union of two rectangles of side lengths l and r, and two semi-disks of radius r,
3. a rounded rectangle, which is the union of the original rectangle, two rectangles of side lengths a and r, two rectangles of side lengths b and r and finally four quarter-disks of radius r.

The areas are respectively πr^2, $2lr + \pi r^2$ and $ab + 2(a+b)r + \pi r^2$. These formulae suggest that there may be a general relationship between the area of the original set and its parallel set at a distance r. It is easy to see that the area of the rounded rectangle $U(R_r)$ can be written :

$$U(R_r) = U(R)r^0 + P(R)r^1 + \pi r^2 \tag{3}$$

where $P(R)$ denotes the boundary length (or perimeter) of the rectangle R. The functions $U(R)$ and $P(R)$ are independent of the scale of the ball $b(0, r)$. We can detect which terms of the equation (3) dominate when r is changed. Intuitively we have

1. terms in r^0 : dense regions (space-filling regions),
2. terms in r^1 : parts that extend along their sides,
3. terms in r^2 : line endings, points of high curvature, sharp tangent discontinuities.

As one last example, let us consider a cube Q with edge length a. Its parallel volume $V_r(Q)$ may be found by decomposition

$$V_r(Q) = a^3 + 6a^2 r + 12a\frac{\pi}{4}r^2 + 8\frac{4\pi}{24}r^3 \tag{4}$$

The $r-$terms are the contributions from respectively the rectangular prisms on the face of Q, from the cylindrical sectors on its edges and from the spherical sectors located on the corners. Again (4) suggests the generalization

$$V_r(Q) = V(Q) + S(Q)r + 2\pi B(Q)r^2 + \frac{4\pi}{3}r^3 \tag{5}$$

where S(Q) is the surface area and $B(Q)$ is the mean breadth.

We can note that there are three possible dilations for a segment E

1. isotropic dilation with a ball of radius r as in Figure 1 (middle) ;
2. normal dilation : dilation of the segment E in the two normal directions to E, as in the $r-$offsetting of E ;
3. tangential dilation : dilation with the segment $[-r, r]$ in the direction of the tangents.

The examples presented above suggest that for simple 3D (2D) geometrical objects, the change in the volume (area) can be computed from the original volume, area, and mean breadth (area and perimeter), as long as the object is inflated or deflated without changing the topology. The convexity of the object is essential for the validity of the previous formulae, which are called Steiner's formulae.

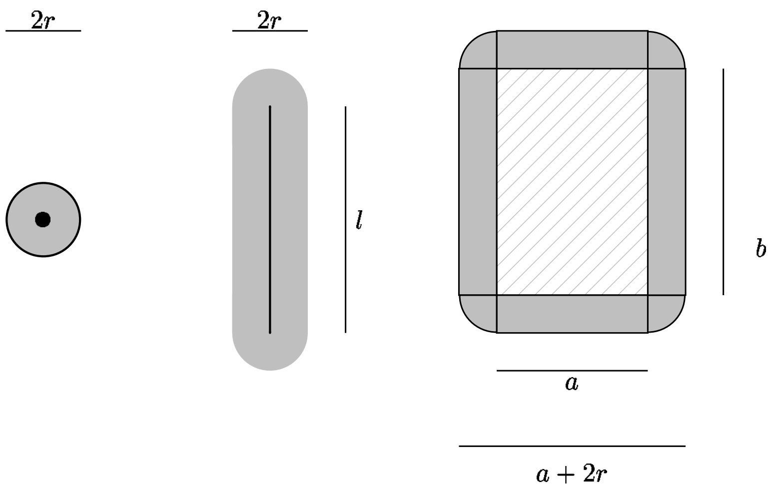

Figure 1. Dilating simple sets with a ball. Three elementary sets are shown with their corresponding dilations : (left) a point, (middle) a segment of length l, (right) a rectangle of side lengths a and b.

3. MINKOWSKI FUNCTIONALS

More generally, Steiner's formula for convex bodies in $d-$dimensional Euclidean space may be written as

$$V_r(K) = \sum_{\nu=0}^{d} \binom{d}{\nu} W_\nu^{(d)}(K)\, r^\nu \tag{6}$$

where the $W_\nu^{(d)}(K)\, r^\nu$ are called Minkowski functionals or curvature integrals. For $d = 3$ one can write

$$V_r(K) = W_0^{(3)}(K)\; + 3W_1^{(3)}(K)\, r^1 + 3W_2^{(3)}(K)\, r^2 + W_3^{(3)}(K)\, r^3 \tag{7}$$

geometric quantity	ν	$W_\nu^{(d)}$	V_ν
V volume	0	V	V
S surface	1	$S/3$	$S/6$
M integral of mean curvature	2	$M/3$	$M/3\pi$
χ Euler characteristic	3	$4\pi\chi/3$	χ

Table 1. Relation between the Minkowski functionals ($W_\nu^{(d)}(K)$ and $V\nu(K)$) of a convex set K and conventional geometrical quantities.

By using identification we can obtain the relation between the Minkowski functionals and the familiar geometrical measures (Table 1).

Thus the Minkowski functionals are generalizations of both the volume and the surface area of a convex body K. Integral Geometry provides us with a complete family of so-called "intrinsic volumes" $W_0^{(d)}(\nu = 0, \ldots, d)$ in $d-$dimensional space, where the usual $V = W_0^{(d)}$ is only one example. Consider the functional $W_2^{(3)}(K)$ and suppose that the boundary ∂K is regular enough to have finite principal curvatures $1/r_1(s)$ and $1/r_2(s)$ at each surface element ds.[9] In this case the surface area $S(K)$ can be written as $\int_{\partial K} \mathrm{ds}$ and the functional $3W_2^{(3)}(K)$ can be represented as the surface integral of the mean curvature $\kappa_1(s)$, which is noted $M(K)$:

$$M(K) = \int_{\partial K} \kappa_1(s)\mathrm{ds} = \int_{\partial K} \frac{1}{2}\left(\frac{1}{r_1(s)} + \frac{1}{r_2(s)}\right)\mathrm{ds}$$

For a ball of radius r, the curvatures are constants, $\kappa_1 = 1/r$ and $\kappa_2 = 1/r^2$, one obtains that $M(b(0,r)) = 4\pi r$.

Now consider a convex object K, which may have edges and vertices where at least one of the principal curvatures is infinite. Then it is useful to introduce the mean breadth, where the mean of breadth is taken over all directions in space. If K is a convex polyhedron then the mean breadth $\bar{b}(K)$ is related to the edge lengths l_i and to the dihedron angles γ_i between the two outer normal vectors to the faces corresponding to the same edge ($i = 1, \ldots, n$). From Santaló[3] we have

$$\bar{b}(K) = \frac{1}{4\pi}\sum_{i=1}^{n} l_i\gamma_i$$

Notice that $M(K) = 2\pi\bar{b}(K)$.

Denote $\mathcal{K}$ as the class of closed bounded convex subsets of $\mathbb{R}^d$ and list some general properties of the functionals $W_\nu^d : \mathcal{K} \to \mathbb{R}$, for $\nu = 0, \ldots, d$.

1. Motion invariance : $W_\nu^d(K) = W_\nu^d(gK)$ (g = rotation plus translation). The Minkowski functionals of a body are independent of its location in space.
2. k-homogeneity : $W_\nu^d(cK) = c^k W_\nu^d(K)$ for $c > 0$ and $k \geq 0$
3. C-Additivity : $W_\nu^d(K_1 \cup K_2) = W_\nu^d(K_1) + W_\nu^d(K_2) - W_\nu^d(K_1 \cap K_2)$.
4. Convex continuity : $W_\nu^d(K_i) \to W_\nu^d(K)$ as $K_i \to K$ (K, K_i convex).
5. Monotonically increasing : $W_\nu^d(K_1) \leq W_\nu^d(K_2)$ if $K_1 \subseteq K_2$

A fundamental result in integral geometry is the completeness of the family of Minkowski functionals. A theorem by Hadwiger[4] states that every motion invariant, $C-$additive and continuous functional $\varphi(K)$ over $\mathcal{K}$ can be written as

$$\varphi(K) = \sum_{\nu=0}^{d} a_\nu W_\nu^{(d)}(K)$$

with suitable coefficients a_ν

4. ISOPERIMETRIC INEQUALITIES AND BLASCHKE DIAGRAM

For many years mathematicians have been interested in inequalities involving geometric functionals of convex figures. The Greeks discovered the first isoperimetic inequality by solving the following problem : "Is it a link between the area A of a plane region and its perimeter L ?". A way to formulate an equivalent statement is in the form of the isoperimetric inequality.

Theorem. Among all regions in the plane, enclosed by a piecewise C^1 boundary curve, with area A and perimeter L,

$$4\pi A \leq L^2 \tag{8}$$

If equality holds, then the region is a circle.

More generally, the isoperimetric problem in $\mathbb{R}^n$ is to minimize the surface area among all domains having given volume, or equivalently, maximize the volume along all domains D whose boundary surfaces have fixed volume. The solutions in both cases is that the unique extremal is the domain bounded by a sphere. The corresponding inequality is

$$n^n \omega_n V(D)^{n-1} \leq V(\partial D)^n$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$ and ∂D is the boundary of D. The equality holds if and only if D is the ball. This inequality is the consequence of the fundamental result established by Brunn and Minkowski. We shall first present this result in case of the plane ($\mathcal{A}(A)$ denotes the area of A).

Theorem. Brunn-Minkowski inequality. Let A, $B \in \mathbb{R}^2$ be arbitrary bounded measurable sets in the plane. Then

$$\sqrt{\mathcal{A}((1-\lambda)A + \lambda B)} \;\geq\; (1-\lambda)\sqrt{\mathcal{A}(A)} + \lambda\sqrt{\mathcal{A}(B)}, \qquad \lambda \in [0,1] \tag{9}$$

This theorem says that since the Minkowski addition tends to "round out" the sets being mixed, the area of the mixed set exceeds the sum of the area of the A and B. We can check this result in the case of rectangles with sides parallel to the axes. The general result can be proved by exhausting the sets A and B by disjoint rectangles. The above theorem can be established in n-dimensional space. If A and B are non-empty compact subsets of $\mathbb{R}^n$, then

$$\mathrm{vol}((1-\lambda)A + \lambda B)^{1/n} \;\geq\; (1-\lambda)\mathrm{vol}(A)^{1/n} + \lambda \mathrm{vol}(B)^{1/n}, \qquad \lambda \in [0,1] \tag{10}$$

The following inequalities of Minkowski can be stated from the previous theorem.

$$S^2 \geq 3VM \tag{11}$$
$$M^2 \geq 4\pi S \tag{12}$$
$$M^3 \geq 48\pi^2 V \tag{13}$$

Blaschke proposed mapping the family of compact convex sets $\mathcal{K}$, in $\mathbb{R}^3$, into a compact region in the plane. In other words, he proposed that each member of $\mathcal{K}$ be mapped to the point (x, y) of $\mathbb{R}^2_+$, where

$$x = \frac{4\pi S}{M^2} \qquad \text{and} \qquad y = \frac{48\pi^2 V}{M^3} \tag{14}$$

The range of this mapping is called now the Blaschke diagram and is denoted by $\overline{\mathcal{K}}$.[10] Relations between three fundamental parameters of a convex body are thus mapped in a plane diagram, as in Figure 2.

The inequalities above can be translated to

$$x^2 \geq y \tag{15}$$
$$1 \geq x \tag{16}$$
$$1 \geq y \tag{17}$$

$$\frac{8}{\pi^2}, \qquad \text{where} \qquad y = 0 \tag{18}$$

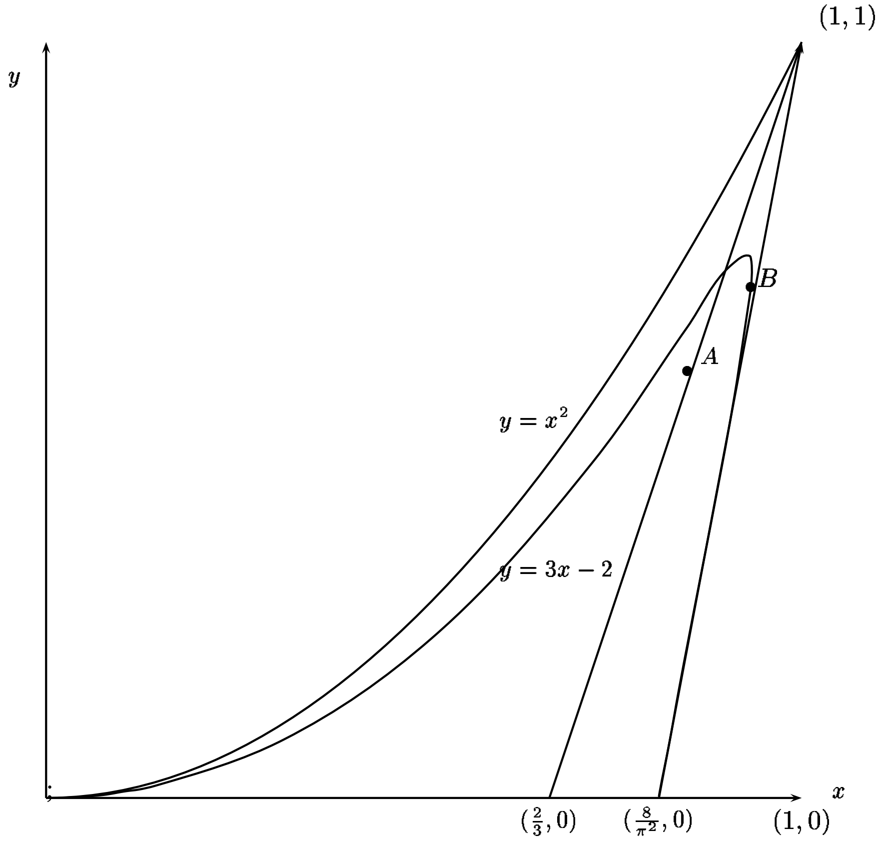

Figure 2. The Blaschke diagram.

Observe that all similar sets are mapped to the same point. Thus the balls are the only bodies mapped into the point $(1, 1)$. The image of points and line segments is in $(0, 0)$. Planar convex sets, for which $V = 0$, are mapped to the $x-$axis, and the disks are the only sets whose image is $(8/\pi^2, 0)$.

Thus $\overline{\mathcal{K}}$ is a bounded region included in the unit square. Let us find the boundary of $\overline{\mathcal{K}}$. The inequalities above assert that $\overline{\mathcal{K}}$.

$$0 \leq x \leq 1, \qquad 0 \leq y \leq x^2 \tag{19}$$

where the parabola $y = x^2$ $(0 \leq x \leq 1)$ belongs to the boundary of $\overline{\mathcal{K}}$

The family of extreme sets mapped to the parabolic boundary are the cap bodies. These are the convex hull of a ball and countably many points exterior to it such that the segments jointing any two of these points intersect the ball. The right boundary is still unknown.

To show that the Blaschke diagram is connected, consider the convex mixing $J = (1 - \lambda)K + \lambda L$ where K and L are elements of $\mathcal{K}$ and $\lambda \in [0, 1]$. Each element of J is a member of $\mathcal{K}$ and it follows from the properties of V, S and M that the image of J is an algebraic curve.[11] In particular, the image of a set K is connected to (1,1) by the curve determined by its outer parallel bodies, that is the bodies of the form $K + rb(0, 1)$. If the image of J lies on the line $y = 3x - 2$, then all its outer parallel bodies are mapped to that line. The point labelled A is the image of the cubes. The point labelled B is the image of the cylinders whose the height is equal to the radius. It is on the curve which is the mapping of cylinders of radius r and of height λr. The volume, surface, mean curvature and Euler characteristic of these previous cylinders are respectively :

$$V = \pi \lambda r^3, \qquad S = 2\pi r^2(1 + \lambda), \qquad M = \pi r(\pi + \lambda), \qquad \chi = 1$$

By varying the parameter λ from zero to infinity, we can change the morphology of the cylinder from prolate to oblate shape.

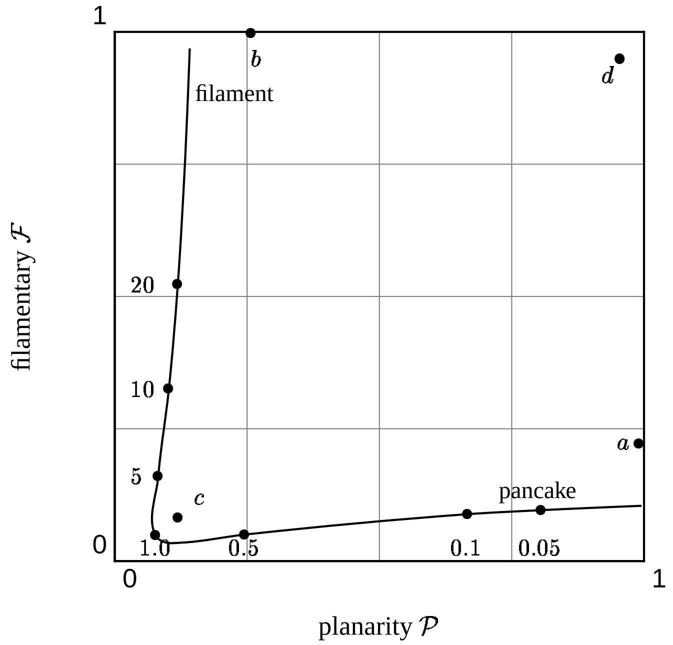

Figure 3. The shape of some convex bodies. The solid line corresponds to cylinders in transition from a filament to a pancake by varying their height.

5. THE SHAPEFINDERS

The convex bodies do not fill the whole unit square. We choose three independent ratios of Minkowski functionals that have dimension of length. Requiring that they yield the radius r applied to a ball, we define

$$\text{Thickness} \quad T_h = \frac{V_0}{2V_1}, \qquad \text{Width} \quad W_i = \frac{2V_1}{\pi V_2}, \qquad \text{Length} \quad L_e = \frac{3V_2}{4V_3}$$

With the isoperimetric inequality, we have $L_e \geq W_i \geq T_h$ for any convex body. Recently Sahni et al.[7] have proposed dimensionless shapefinders by

$$\text{Planarity} \quad \mathcal{P} = \frac{W_i - T_h}{W_i - T_h}, \qquad \text{Filamentary} \quad \mathcal{F} = \frac{L_e - W_i}{L_e + W_i} \tag{20}$$

Now consider the example of parallelipeds. An ideal pancake (having vanishing thickness) has one characteristic dimension much smaller than the remaining two, so that $T_h \ll W_i \simeq L_e$ and $K_{\mathcal{P},\mathcal{F}} \simeq (1,0)$. An ideal filament (a one dimensional object) has two characteristic dimensions much smaller than the third so that $T_h \simeq W_i \ll L_e$ and $K_{\mathcal{P},\mathcal{F}} \simeq (0,1)$. All three dimensions of a cube are equal so that $T_h \simeq W_i \simeq L_e$ and $K_{\mathcal{P},\mathcal{F}} \simeq (0,0)$. Finaly consider a ribbon for which $T_h \ll W_i \ll L_e$ and $K_{\mathcal{P},\mathcal{F}} \simeq (1,1)$. These four paralleliped correspond to the points labelled respectively a, b, c and d on the Figure 3.

a, b, c	Morphology	$(\mathcal{P}, \mathcal{F})$	Thickness	Width	Length
(10 000, 10 000, 1)	Pancake	$(0.990, 0.221)$	1.497	318,782	500.250
(10 000, 1, 1)	Filament	$(0.258, 0.998)$	0.749	1.273	2500.000
(10 000, 100, 1)	Ribbon	$(0.954, 0.950)$	1.485	63.661	2525.250
(10 000, 10 000, 10 000)	Cube	$(0.120, 0.0817)$	5 000.000	6366.197	7500.000

Table 2. Shapefinders for a paralleliped with side lengths a, b, c. T_h, W_i and L_e have dimension of length, $\mathcal{P}$ and $\mathcal{F}$ are dimensionless.

6. CONCLUSION

In this paper, we have presented the Blaschke diagram and one variant which have allowed us to represent 3D convex bodies with a point in $\mathbb{R}^2$. The major limit of the mapping is that it is restricted to convex bodies. Indeed, both convex and non-convex bodies can be represented with the same point. It will be then essential to use information of semantic type in order to discriminate between objects.

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